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Cosmic Microwave Background (Cont'd):

We now perform a more careful analysis of perturbations in a fluid.

Jeans Analysis in a Fluid:

We start by considering a perfect fluid that is perturbed around a stationary background (i.e., no expansion). Being a continuous medium, the fluid is described by its density $\rho(\vec{r},t)$ and velocity $\vec{v}(\vec{r},t)$. We note that pressure is related to the density via the equation of state $P(\vec{r},t) = \omega\,\rho(\vec{r},t)$.

The relevant equations that govern evolution of the fluid in the presence of gravity are the following:

$$\begin{cases} \dfrac{\partial \rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{v}) = 0 & \text{Continuity equation} \\ \dfrac{\partial \vec{v}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{v} + \dfrac{1}{\rho}\vec{\nabla}P + \vec{\nabla}\Phi = 0 & \text{Euler equation} \\ \nabla^2 \Phi = 4\pi G\rho & \text{Poisson equation} \end{cases} \qquad (I)$$

Now consider a slightly perturbed fluid where $\rho = \rho_0 + \rho_1(\vec{r},t)$, $\Phi =$

$\Phi_0 + \Phi_1(\vec{r},t)$, and $\vec{v} = \vec{v}_0 + \vec{v}_1(\vec{r},t)$. Here $\rho_0, \Phi_0, \vec{v}_0$ are the background values that are taken to be constant, and $\rho_1, \Phi_1, \vec{v}_1$ denote small perturbations ($\rho_1 \ll \rho_0$, $\Phi_1 \ll \Phi_0$, $|\vec{v}_1| \ll |\vec{v}_0|$). Perturbations in the pressure are found from the equation of state $P = P_0 + P_1(\vec{r},t)$, where $P_0 = w\rho_0$ and $P_1 = w\rho_1$.

After using the fact that the background values obey equations in (I) and keeping terms to the linear order in perturbations, we find:

$$\begin{cases} \dfrac{\partial \rho_1}{\partial t} + \rho_0 \vec{\nabla} \cdot \vec{v}_1 = 0 \\ \dfrac{\partial \vec{v}_1}{\partial t} + \dfrac{v_s^2}{\rho_0} \vec{\nabla} \rho_1 + \vec{\nabla} \Phi_1 = 0 \qquad (v_s^2 \equiv w) \qquad (\text{II}) \\ \nabla^2 \Phi_1 = 4\pi G \rho_1 \end{cases}$$

By taking the partial derivative of both sides of the first equation above with respect to time, and using the other equations in (II), we arrive at a wave equation for ρ_1:

$$\frac{\partial^2 \rho_1}{\partial t^2} - v_s^2 \nabla^2 \rho_1 = 4\pi G \rho_0 \rho_1$$

For a single perturbation mode with wavenumber k we have:

$$\rho_1 = \delta_K \rho_0 \exp(i\vec{k}\cdot\vec{r} - i\omega t)$$

This results in:

$$\ddot{\delta}_K + (v_s^2 K^2 - 4\pi G\rho_0)\,\delta_K = 0 \qquad (\text{III})$$

Here $\delta_K \equiv \frac{\rho_1}{\rho_0}$. The frequency of oscillations ω is given by:

$$\omega^2 = v_s^2 (k^2 - k_J^2) \qquad k_J \equiv \left(\frac{4\pi G\rho_0}{v_s^2}\right)^{\frac{1}{2}}$$

k_J is called the "Jeans wavenumber" and $\lambda_J = \frac{2\pi}{k_J}$ is the "Jeans wavelength". For $k > k_J$, we have $\omega^2 > 0$, which implies stable oscillations. On the other hand, $\omega^2 < 0$ for $k < k_J$, which results in unstability and exponential growth of perturbation amplitude.

The above analysis can be extended to an expanding homogeneous and isotropic universe. In this case:

$$\rho_0(t) = \rho_0(t_0)\left[\frac{a(t)}{a(t_0)}\right]^3, \qquad \vec{v}_0 = \frac{\dot{a}(t)}{a(t)}\vec{r}, \qquad \vec{\nabla}\Phi_0 = \frac{4\pi G\rho_0}{3}\vec{r}$$

This leads to:

$$\ddot{\delta}_K + 2H(t)\,\dot{\delta}_K + \left[\frac{v_s^2 k^2}{a(t)^2} - 4\pi G\rho_0(t)\right]\delta_K = 0 \qquad (\text{IV})$$

We notice two differences with Eq. (II). First, there is a damping term $2H(t)\dot{\delta}_k$, which is due to expansion of the universe. Second, the frequency is now time dependent, which is again because of the expansion.

A closer look at Eq. (IV) reveals that mode initially start in the unstable mode upon entering the horizon $\frac{k}{a} < k_J$. However, both the physical wavenumber $\frac{k}{a}$ and the Jeans wavenumber k_J are functions of time in an expanding universe. The important point is that k and k_J both decrease with time, but this happens at a faster rate for k_J. In a radiation dominated universe $k \propto t^{-\frac{1}{2}}$ and $k_J \propto t^{-1}$. Similarly, in a matter dominated univer $k \propto t^{-\frac{2}{3}}$ and $k_J \propto t^{-1}$. This implies that modes that are initially unstable, and grow because of the gravitational attraction, will eventually become stable and undergo oscillations. The larger v_s is, the smaller k_J will be, which is intuitively expected because larger v_s is equivalent to a higher pressure. Moreover, oscillations in an expanding universe are inevitably damped because of the expansion, which is governed by the term $2H_{(t)}\dot{\delta}_k$ in Eq. (IV).

So far, we have considered a single-component fluid. In general, however, the fluid has multiple components. For example, in the early universe (for $t > 1\,sec$) the content of the universe is represented by a fluid with three components: photons, baryons, and dark matter. For a multi-component fluid we can find an equation that describes evolution of perturbation for each component:

$$\rho_0 = \sum_i \rho_{0,i} \qquad \delta_{k,i} \equiv \frac{\delta\rho_{0,i}}{\rho_{0,i}} \quad , \quad \epsilon_i \equiv \frac{\rho_{0,i}}{\rho_0}$$

Focusing on the non-relativistic species, the equation for $\delta_{k,i}$ is:

$$\ddot{\delta}_{k,i} + 2H_{(t)}\dot{\delta}_{k,i} + \left[\frac{v_{s,i}^2 k^2 \delta_{k,i}}{a_{(t)}} - 4\pi G \rho_{0(t)} \sum_j \epsilon_j \delta_{k,j}\right] = 0 \qquad (V)$$

We note that the term from pressure depends on the speed of sound for the component under consideration, while the term from gravity is the sum of contributions from all (non-relativistic) components. This because of the universality of the gravity. We will use Eq. (V) to study perturbations in the baryons and dark matter and their effect on the CMB spectrum next.